sponding media components. The quantities $K_{\alpha\beta}$, $K^j_{\alpha\beta}$, $K^{jq}_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) are then the dielectric permeabilities of the material in the corresponding components of the medium. The model homogeneous anisotropic medium corresponding to the structure is governed by the law (5.5), where K_{11}, K_{12}, K_{22} are the dielectric permeabilities of the matrix material. If the coupling media (or only some of them) are isotropic, then all our results remain valid. We need only set $K_{12} = 0$, $K_{11} = K_{22} = K$ for each corresponding domain.

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CORRELATION FUNCTIONS OF THE ELASTIC FIELD OF MULTIPHASE POLYCRYSTALS

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An approximation of the homogeneity of a linear combination of the stresses and strains $\sigma + b\varepsilon = \text{const}$ is proposed to evaluate the correlation functions of the elastic field of micro-inhomogeneous media. This approximation is a generalisation of the Voigt and Reuss hypotheses according to which the strains ε and the stresses σ are considered homogeneous, respectively. Independence of the spatial fluctuations of the volume and shear components of the elastic field holds within the scope of the approximation made. It is shown that the proposed relationship is satisfied exactly for laminar materials, but approximately for fibrous and granular materials. An explicit form is found for the tensor b in the singular approximation of random function theory under the assumption of isotropy of the properties of each of the fibrous and granular material phases and the correlation functions and stress and strain fields dispersions are calculated. It is shown that in this approximation the coordinate and tensor dependences of the correlation functions of the stress and strain fields are separated. An analogous computation is performed for multiphase polycrystals in the correlation approximation according to which correlation functions of elastic moduli of not higher than the second order are taken into account. In this approximation, the coordinate and tensor dependences of the correlation functions of the elastic field do not separate. Conditions are found under which the correlation approximation results in independence of the volume and shear components of the elastic field fluctuations.

The exact computation of the stress and strain fields is a complex problem for the deformation of micro-inhomogeneous media (composite materials, single-and multiphase polycrystals, etc.). The difficulty is due to the need to take account of the interaction between all the inhomogeneity elements, which reduces to evaluating multiple integrals of certain functions including multipoint correlation or structural functions. To overcome this difficulty we can use different approximate methods. The simplest were proposed by Voigt [1] and Reuss [2] in connection with the problem of evaluating the effective elastic moduli of polycrystals. According to these hypotheses, it is assumed that homogeneity of the stresses (Reuss approximation) $\sigma = \langle\sigma\rangle = \text{const}$, or the strains (Voigt approximation) $\varepsilon = \langle\varepsilon\rangle = \text{const}$ holds. The angular brackets here denote the statistical average over the ensemble of monotypic situations, and σ and ε are second rank tensors.

The authors [3] proposed a singular approximation of the renormalization method equivalent to the hypothesis [4] of strong isotropy. In this approximation only the singular components of the second derivatives of the Green's tensor of the equilibrium equation are taken into account, which means "spreading" of the elastic field over the grain of the inhomogeneity. On the other hand, the approximation of the homogeneous field within the phase limits is fundamental in the variational method of evaluating the effective elastic moduli. Such an approach is taken implicitly in the self-consistency method [5].

1. Let us generalize the Voigt and Reuss approximation by demanding the homogeneity of some combinations of the stress and strain fields

$$\boldsymbol{\sigma} + \mathbf{b}\boldsymbol{\varepsilon} = \langle \boldsymbol{\sigma} + \mathbf{b}\boldsymbol{\varepsilon} \rangle = \text{const} \tag{1.1}$$

From (1.1) there follows

$$\boldsymbol{\sigma}' = -\mathbf{b}\boldsymbol{\varepsilon}', \qquad \boldsymbol{\varepsilon}' = -\mathbf{a}\boldsymbol{\sigma}', \qquad \mathbf{ab} = \mathbf{I} \qquad (\mathbf{x}' \equiv \mathbf{x} - \langle \mathbf{x} \rangle) \tag{1.2}$$

i.e. the random stress and strain field components are interrelated by using some constant fourth rank tensor $\mathbf{b}$ or its inverse tensor $\mathbf{a}$. Here $\mathbf{I}$ is the unit fourth rank tensor.

A relationship analogous to (1.2) was proposed as a hypothesis [5, 6] for the evaluation of the effective elastic moduli of single-phase polycrystals of cubic symmetry. It was also used in [7] to construct a theory of plasticity of polycrystals.

A relation between the multipoint correlation functions of the stress and strain fields

$$\langle \boldsymbol{\sigma}'(\mathbf{r}_1) \otimes \boldsymbol{\sigma}'(\mathbf{r}_2) \otimes \ldots \otimes \boldsymbol{\sigma}'(\mathbf{r}_n) \rangle = (-1)^n \langle \mathbf{b}\boldsymbol{\varepsilon}'(\mathbf{r}_1) \otimes \mathbf{b}\boldsymbol{\varepsilon}'(\mathbf{r}_2) \otimes \ldots \otimes \mathbf{b}\boldsymbol{\varepsilon}'(\mathbf{r}_n) \rangle \tag{1.3}$$

or between the single-point moments

$$\langle [\boldsymbol{\sigma}'(\mathbf{r}_1) \otimes]^n \rangle = (-1)^n \langle [\mathbf{b}\boldsymbol{\varepsilon}'(\mathbf{r}_1) \otimes]^n \rangle \tag{1.4}$$

can be found within the scope of the approximation (1.2). Here and henceforth $\mathbf{A} \otimes \mathbf{B}$ denotes the direct product whose rank equals the sum of the ranks of the factors, $\mathbf{AB}$ is the product in which convolution of the inner subscripts is made. The rank of such a product equals the difference between the ranks of the factors if the ranks of $\mathbf{A}$ and $\mathbf{B}$ do not agree, and the rank of one of the factors otherwise. Finally, $\mathbf{A} \cdot \mathbf{B}$ will denote the product with complete convolution of all the subscripts transforming two tensors of identical rank into a scalar.

Let us note that for laminar materials comprised of arbitrary anisotropic components, the relationships (1.1) and (1.2) are satisfied exactly, as results from the conditions

$$\varepsilon_{\alpha\beta} = \langle\varepsilon_{\alpha\beta}\rangle, \qquad \sigma_{i3} = \langle\sigma_{i3}\rangle, \qquad \alpha,\ \beta = 1,\ 2 \tag{1.5}$$

Hence $b_{i3kl} = 0$ and $a_{\alpha\beta kl} = 0$, i.e. some components of the tensor $\mathbf{b}$ vanish, while others become infinite. Here the x_3-axis is directed orthogonally to the layers. If the material is of arbitrary structure, then (1.1) and (1.2) are satisfied only approximately.

For a granular medium which is a quasi-isotropic mixture of anisotropic or isotropic components in a singular approximation, (1.1) and (1.2) are satisfied and the tensor $\mathbf{b}$ is [8]

$$\mathbf{b}^c = 3b_K{}^c\mathbf{V} + 2b_\mu{}^c\mathbf{D}, \qquad \mathbf{V}+\mathbf{D} \equiv \mathbf{I}, \qquad 3b_K{}^c = 4\mu_c \tag{1.6}$$

$$6b_\mu{}^c = \mu_c\,\frac{9K_c+8\mu_c}{K_c+2\mu_c}$$

where $\mathbf{V}$ and $\mathbf{D}$ are the volume and deviator components of the unit tensor $\mathbf{I}$, and the subscript c denotes that these quantities refer to a homogeneous comparison body which is isotropic for untextured granular media. The evaluation of the elasticity constants of the comparison body according to the known elasticity constants of the components can be performed by the method proposed in [9], say.

For fibrous materials we find analogously

$$a_{11}+a_{12} = \frac{1}{2}a_{44} = \frac{1}{2\mu_c}, \quad 2(a_{11}-a_{12}) = a_{66} = \frac{1}{\mu_c}\,\frac{3K_c+7\mu_c}{3K_c+\mu_c} \tag{1.7}$$

$$a_{13} = a_{33} = 0$$

Therefore, (1.1) and (1.2) are satisfied in the singular approximation for both granular and fibrous structures.

2. Relations between the stress and strain correlation functions can be calculated by using (1.3) in the known tensors $\mathbf{a}$ and $\mathbf{b}$. If the tensor $\mathbf{b}$ is given by (1.6), then the binary stress and strain correlation functions are of the form

$$S_{ijkl}(r) = D^q_{ijkl}\varphi(\mathbf{r}), \qquad E_{ijkl}(r) = D^p_{ijkl}\varphi(\mathbf{r}) \tag{2.1}$$

$$D^x_{ijkl} \equiv (x_{ij}-\langle x_{ij}\rangle)(x_{kl}-\langle x_{kl}\rangle)$$

$$q_{ij} \equiv \frac{b_K{}^c K'\varepsilon}{K_c+b_K{}^c}\delta_{ij} + \frac{2b_\mu{}^c\mu'}{\mu_c+b_\mu{}^c}e_{ij} \tag{2.2}$$

$$p_{ij} \equiv \frac{K'\varepsilon}{3(K_c+b_K{}^c)}\delta_{ij} + \frac{\mu'}{\mu_c+b_\mu{}^c}e_{ij}$$

The relationship (2.1) can be considered as a parametric form of the relation between the functions S_{ijkl} and E_{ijkl}.

We hence find for the root-mean-square fluctuations of the volume and shear stresses and strains

$$\sigma_* = 3b_K{}^c\varepsilon_*, \qquad s_* = 2b_\mu{}^c e_* \tag{2.3}$$

$$\sigma_* \equiv 3\mathbf{S}\cdot\mathbf{V}, \quad \varepsilon_*{}^2 \equiv 3\mathbf{E}\cdot\mathbf{V}, \quad s_*{}^2 \equiv \mathbf{D}\cdot\mathbf{S}, \quad e_*{}^2 \equiv \mathbf{D}\cdot\mathbf{E} \tag{2.4}$$

It is seen that the relationships $\sigma_* - \varepsilon_*$ and $s_* - e_*$ are outwardly analogous to the Hooke's law for the volume and deviator components of the stresses and strains. Therefore, independence of the volume and shear fluctuations of the stress and strain fields holds for a granular medium in the singular approximation.

3. With the limits of an inhomogeneity element the elastic field is considered homogeneous in the singular approximation. Hence, it is impossible to describe such forms of grain deformation as flexure and twist in the singular approximation. In this connection, it is interesting to turn to the correlation approximation which possesses the advantage that field homogeneity within a given grain is not assumed here although it indeed results in a wider bracket for effective elastic moduli.

The correlation approximation has been used to evaluate the connection between the root-mean-square fluctuations of the stress and strain fields for isotropic macro-deformations in [10]. The general case applied to composites from isotropic phases has been examined in [11].

The appropriate computation for multiphase polycrystals is presented below in the correlation approximation. The coordinate and tensor dependences of the correlation functions of the elastic fields do not separate in this approximation.

Performing the calculation analogously to that done earlier in [11], we obtain

$$\mu_c{}^2 E_{ijkl}(\mathbf{r}) = Q^{ij}_{kl}\,[\delta_{ip}\delta_{kq}J_{jlrs} + 2\varkappa\delta_{ip}J_{jklqrs} + \varkappa^2 J_{ijklpqrs}]\,D^{\lambda}_{prqs} \qquad (3.1)$$

$$S_{ijkl}(\mathbf{r}) = Q^{ij}_{kl}\,[D^{\lambda}_{ijkl}\varphi + 2\lambda^{c}_{ijpq}\mu_c^{-1}(D^{\lambda}_{klpr}J_{qr} + \varkappa D^{\lambda}_{klrs}J_{pqrs}) + \lambda^{c}_{ijpq}\lambda^{c}_{klrs}E_{pqrs}] \qquad (3.2)$$

$$\lambda_{ij} \equiv \lambda_{ijkl}\,\langle\varepsilon_{kl}\rangle, \qquad \varkappa \equiv \frac{3K_c + \mu_c}{3K_c + 4\mu_c}$$

Here the quantities D^{α}_{ijkl} are given by (2.2), and the commutation operator $Q_{kl}{}^{ij}$ and the tensor $J_{ij\ldots}$ are determined in [11]. For $r = 0$ the expressions (3.1) yield the central second order moments of the strain and stress tensors. The corresponding expressions can be represented as

$$15\mu_c{}^2 \overset{\circ}{E}_{ijkl} = {}^1/_{63}\,\varkappa^2\Lambda\delta_{ijkl} - {}^1/_7\,\varkappa(1 - {}^4/_9\,\varkappa)\,\Lambda^{(2)}_{1ijkl} + (1 - {}^4/_7\,\varkappa + {}^8/_{63}\,\varkappa^2)\,\Lambda_{ijkl} + Q^{ij}_{kl}\,(\delta_{ik}\Lambda_{jplp} - \Lambda_{ikjl}) \qquad (3.3)$$

$$15\overset{\circ}{S}_{ijkl} = 60\mu_c{}^2\overset{\circ}{E}_{ijkl} + (1 - 2\varkappa)(1 - {}^{10}/_7\,\varkappa)\,\Lambda\delta_{ij}\delta_{kl} + (8\varkappa - 5)(D^{\lambda}_{ijkl} - 2\delta_{[ij}D^{\lambda}_{kl]\,pp}) + 4(2\varkappa - 1)(1 - {}^4/_7\,\varkappa)\,\delta_{[ij}\Lambda_{kl]} \qquad (3.4)$$

$$\Lambda_{ijkl} \equiv D^{\lambda}_{ijkl} + D^{\lambda}_{ikjl} + D^{\lambda}_{iljk}; \quad \Lambda \equiv \Lambda_{iikk} = D^{\lambda}_{iikk} + 2D^{\lambda}_{ikik} \qquad (3.5)$$

$$\Lambda^{(2)}_{1ijkl} \equiv \sum_{P}\delta_{ij}\Lambda_{kl}, \qquad \Lambda_{kl} \equiv \Lambda_{iikl}$$

The summation in (3.5) is over all distinct commutations of the subscripts, i.e. the quantity $\Lambda^{(2)}_{1ijkl}$ contains six terms. The square brackets denote the operation of symmetrization over subscript pairs. The volume and deviator convolutions of the central second order moments can be found from (3.3) and (3.4). Introducing the functions σ_*, s_*, ε_* and e_* by using (2.4), we obtain

$$\begin{aligned} 15\mu_c{}^2\varepsilon_*{}^2 &= (1 - \varkappa)^2\Lambda \\ 45\mu_c{}^2 e_*{}^2 &= 2(1 - \varkappa)^2\Lambda + {}^3/_2\,(3D^{\lambda}_{pqpq} - D^{\lambda}_{ppqq}) \\ 15\sigma_*{}^2 &= (4\varkappa - 1)^2\,\Lambda + 5(5 - 8\varkappa)\,D^{\lambda}_{iikk} \\ 45 s_*{}^2 &= 8(1 - \varkappa)^2\Lambda + (8\varkappa + 1)(3D^{\lambda}_{pqpq} - D^{\lambda}_{ppqq}) \end{aligned} \qquad (3.6)$$

It follows from (3.6) that each of the four desired quantities is expressed in terms of the two convolutions D^{λ}_{pqpq} and D^{λ}_{ppqq}. Eliminating them, we find the connection between the stresses and strains

$$\begin{aligned} \sigma_*^2 &= 4\mu_c^2\,[{}^2/_3\,(11-8\varkappa)\,\varepsilon_*^2 + (8\varkappa-5)\,e_*^2] \\ s_*^2 &= {}^4/_9\,\mu_c^2\,[(5-8\varkappa)\,\varepsilon_*^2 + {}^3/_2\,(8\varkappa+1)\,e_*^2] \end{aligned} \qquad (3.7)$$

According to (1.2), the volume and shear components of the stress and strain field fluctuations should be independent. It is seen from (3.7) that the independence noted will hold only upon compliance with the condition $3K_c = 4\mu_c$, i.e. for $\varkappa = {}^5/_8$. In this particular case

$$\sigma_* = 4\mu_c\varepsilon_*, \qquad s_* = 2\mu_c e_* \qquad (3.8)$$

The equalities (3.8) agree with (2.3) since $b_K{}^c = {}^4/_3\,\mu_c$ according to (1.6), and $b_\mu{}^c = \mu_c$ for $\varkappa = {}^5/_8$.

Therefore, (1.1) and (1.2) are only satisfied for $K_c = {}^4/_3\,\mu_c$ in the correlation approximation. For an arbitrary relationship between the volume of shear moduli of the material, the fluctuation in the volume component of the stress field is expressed not only in terms of the volume strain fluctuations but also in terms of the shear strain fluctuations, and conversely.

Comparing the correlation and singular approximations, we note that the effective elastic moduli evaluated by using the singular approximation turn out to be nearer to the exact value than those computed by using the correlation approximation. On the other hand, as has already been noted, the singular approximation possesses the disadvantage that it does not permit computation of the grain rotations.

In fact, the correlation tensor of the angle of grain rotation is defined by the expression

$$\Omega_{ij} = -{}^1/_4 e_{ipq} e_{jrs} U_{pr,qs}.$$

In a singular approximation the distortion tensor is [8]

$$u_{i,j} = u^c_{i,j} + G_{ik,jl} * \lambda'_{klpq}\varepsilon_{pq}$$

Here $u_{i,j}{}^c$ is the distortion tensor of the comparison field, G_{ik} is the Green's tensor of the equilibrium equation, and e_{ipq} is the unit antisymmetric Levi-Civita tensor.

In the singular approximation we have [3]

$$G^s_{ik,jl} = -\frac{1}{3\mu^c}\left[\delta_{ik}\delta_{ej} - \frac{\varkappa}{5}\,\delta_{ijkl}\right]\delta(\mathbf{r})$$

Hence, it is seen that convolution of the tensor $G^s_{ik,jl}$ with the tensor $\lambda'_{klpq}\varepsilon_{pq}$ which is symmetric relative to the subscripts kl will yield a tensor symmetric in the subscripts ij. Therefore, the vector of the angles of rotation $\omega_i = -{}^1/_2\,e_{ijk}u_{j,k}$ can be different from zero only because of the comparison field $u_{i,j}$ which is always regular. Hence, the central moment function of the angle of rotation vector Ω_{ij} vanishes. At the same time $\Omega_{ij} \neq 0$ in the correlation approximation, with the exception of purely volume macro-deformation when $\langle\varepsilon_{ij}\rangle = {}^1/_3\,\langle\varepsilon\rangle\delta_{ij}$.

Upon using higher approximations of the theory of random functions, the quantity $\mathbf{l}$ in (1.1) and (1.2) is no longer a constant tensor, and becomes a matrix integral operator whose multiplicity is defined by the approximation used. In this case the approximation of homogeneity of a certain combination of the stress and strain fields is evidently meaningless.

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ON THE IRREDUCIBILITY OF THE EQUATIONS OF THE RESTRICTED CIRCULAR, THREE-BODY PROBLEM TO THE STÄCKEL TYPE EQUATIONS

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It is shown that no generalized coordinates exist in which the total integral of the Hamilton-Jacobi equation for a restricted circular, three-body problem can be represented in the form of a finite sum of the functions each of which depends on a single generalized coordinate.

The Hamilton-Jacobi equation for a restricted plane circular, three-body problem in elliptical variables u, v has the form [1]

$$\left(\frac{\partial S}{\partial u}\right)^2 + \left(\frac{\partial S}{\partial v}\right)^2 + \frac{\partial S}{\partial u}\left[nc^2 \sin 2v - nc(a_1 - a_0)\,\mathrm{ch}\, u \sin v\right] + \frac{\partial S}{\partial v}\left[nc^2\, \mathrm{sh}\, 2u - nc(a_1 - a_0)\,\mathrm{sh}\, u \cos v\right] = F_1(u) + F_2(v) \tag{1}$$